SOLUTION OF THE PROBLEM OF THE DIFFRACTION OF AN ACOUSTIC WAVE BY A CONE

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The problem of the diffraction of a plane unsteady acoustic wave by an infinite circular cone of arbitrary vertex angle was considered in [1 and 2] where numerical and asymptotic methods were used. In the present paper an analytic solution is obtained in the axisymmetrical case through the use of integral tranforms. The solution is further studied in the case of a step-function pressure wave, and the results of numerical computations are given.

1. Suppose an acoustic pressure wave

$$u_0 = (c_0 t - r \cos\theta)^\nu \, \eta\,(c_0 t - r\cos\theta)$$

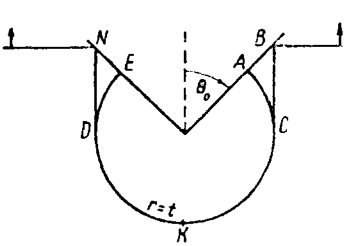

Fig. 1

is incident on a cone (Fig. 1) of arbitrary angle $2\theta_0$. Here $\eta(x)$ is the Heaviside function, c_0 is the velocity of sound in the medium, r and θ are spherical coordinates of a point; and $\operatorname{Re}\nu > -1$. The axis of the cone is perpendicular to the incident wave front, which touches the vertex of the cone at the instant $t = 0$. With no loss of generality we may set $c_0 = 1$ and seek a solution in the form $u = w + u_0$. Then the equation of the perturbed motion of the medium and the boundary and initial conditions assume the form

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial w}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial w}{\partial\theta}\right) = r^2\frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial w}{\partial\theta} = -\frac{\partial u_0}{\partial\theta} \quad \text{at} \quad \theta = \theta_0, \qquad w = \frac{\partial w}{\partial t} = 0 \quad \text{at} \quad t = 0 \tag{1.1}$$

(In the case when the derivative of u_0 does not exist in the ordinary sense, it is to be understood as a generalized derivative.) It is assumed that the function $w(r, \theta, t)$ satisfies the conditions under which a Laplace tranform in t is applicable, and that its tranform admits a Kontorovich-Lebedev tranform in r [3].

2. To the system (1.1) we first apply a Laplace transform in t, and then a Kontorovich-Lebedev tranform in r. Then the equation and boundary condition become

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\psi}{d\theta}\right) + \left(\mu^2 - \frac{1}{4}\right)\psi = 0 \qquad \left(\psi = \int_0^\infty \frac{\Phi(r,\theta,p)}{\sqrt{r}} K_\mu(pr)\,dr\right) \tag{2.1}$$

$$\frac{d\psi}{d\theta} = -\frac{\pi^{3/2}\,\Gamma(1+\nu)}{\sqrt{2}\,p^{3/2+\nu}\cos\pi\mu} P^{1}_{\mu-1/2}(\cos\theta_0) \qquad \text{at} \quad \theta = \theta_0 \tag{2.2}$$

$$\operatorname{Re} p > a_0 \geqslant 0, \quad \operatorname{Re}\mu = 0, \quad \Phi(r,\theta,p) \div w(r,\theta,t)$$

Here $\Gamma(x)$ is the Gamma function; $P_{\mu}^{a}(a)$ are the associated Legendre functions of the first kind for the interval $(-1, 1)$ [4]; and $K_{\mu}(x)$ is the MacDonald function.

The solution of (2.1) which has no singularity in the region $\theta_0 \leq \theta \leq \pi$ and satisfies boundary condition (2.2) may be written in the following form:

$$\psi = \frac{\pi^{3/2}\Gamma(1+\nu)\, P^{1}_{\mu-1/2}(\cos\theta_0)\, P_{\mu-1/2}(-\cos\theta)}{\sqrt{2}\, p^{3/2+\nu}\cos\pi\mu\, P^{1}_{\mu-1/2}(-\cos\theta_0)} \tag{2.3}$$

Applying an inverse Kontorovich-Lebedev transform in μ, we obtain

$$\Phi = \frac{\sqrt{\pi}\,\Gamma(1+\nu)}{\sqrt{2r}\, p^{3/2+\nu}\, i}\int_{-i\infty}^{i\infty}\frac{P^{1}_{\mu-1/2}(\cos\theta_0)\, P_{\mu-1/2}(-\cos\theta)}{\cos\pi\mu\, P^{1}_{\mu-1/2}(-\cos\theta_0)}\, I_{\mu}(pr)\,\mu\, d\mu \tag{2.4}$$

Here $I_{\mu}(x)$ is the modified Bessel function of the first kind.

The integral on the right-hand side of (2.4) converges and represents an analytic function in the region $|\arg p| < \tfrac{1}{2}\pi - (2\theta_0 - \theta)$. But the function Φ, as Laplace transform of the desired solution, should be analytic in the region $\operatorname{Re} p > a_0$. We use an analytic continuation of the right-hand side of (2.4) to define Φ as an analytic function in the entire region $\operatorname{Re} p > a_0$ for any θ in the interval $\theta_0 \leq \theta < \pi$.

Since

$$P^{m}_{\mu-1/2}(\cos\theta) \sim \left(\frac{2}{\pi\sin\theta}\right)^{1/2}\mu^{m-1/2}\sin\left(\mu\theta + \frac{m\pi}{2} + \frac{\pi}{4}\right) \qquad (\varepsilon \leqslant \theta \leqslant \pi - \varepsilon,\ \varepsilon > 0)$$

$$I_{\mu}(pr) \sim \frac{1}{\Gamma(1+\mu)}\left(\frac{pr}{2}\right)^{\mu} \qquad \text{as } |\mu| \to \infty$$

the integrand in (2.4) decays rapidly in the right half-plane and the contour of integration may be deformed to the contour L (Fig. 2). This is because the integrand in (2.4) is analytic in the half-plane $\operatorname{Re}\mu > 0$ everywhere outside the real axis. As is evident from Fig. 2, the contour L proceeds to infinity along the real axis, circumventing each simple pole of the integrand. It is now easy to see that the integral in (2.4) (but now taken along L) converges and represents an analytic function in the half-plane $\operatorname{Re} p > 0$; thus by the uniqueness theorem it represents the function Φ in the region $\operatorname{Re} p > a_0$.

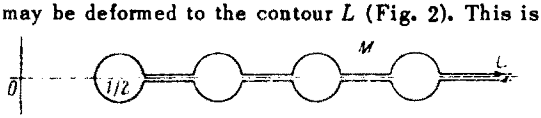

Fig. 2

If we now apply an inverse Laplace transformation to (2.4) and interchange the order of integration (which interchange may be justified by a detailed argument), we obtain

$$w = -\frac{\Gamma(1+\nu)}{2\sqrt{2\pi r}}\int_L \frac{P^{1}_{\mu-1/2}(\cos\theta_0)\, P_{\mu-1/2}(-\cos\theta)\,\mu}{\cos\pi\mu\, P^{1}_{\mu-1/2}(-\cos\theta_0)}\, d\mu \int_{a-i\infty}^{a+i\infty}\frac{I_{\mu}(pr)}{p^{3/2+\nu}}\, e^{pt}dp \qquad (a > 0) \tag{2.5}$$

The integral with respect to p in (2.5) is calculated in Section 4.

3. If one uses Formula (4.2) (see Section 4), then expression (2.5) for $w(r, \theta, t)$ may written in the following form:

$$w = \frac{\Gamma(1+\nu)}{2ri}(r^2 - t^2)^{1/2(1+\nu)}\int_L \frac{P^{1}_{\mu-1/2}(\cos\theta_0)\, P_{\mu-1/2}(-\cos\theta)\, P^{-1-\nu}_{\mu-1/2}(-t/r)\,\mu d\mu}{\cos\pi\mu\, P^{1}_{\mu-1/2}(-\cos\theta_0)} \qquad (-r < t < r)$$

$$w = \frac{\Gamma(1+\nu)}{\pi r i}e^{\pi i\nu}(t^2 - r^2)^{1/2(1+\nu)}\int_L \frac{\cos[(\mu-\nu)\pi]\, P^{1}_{\mu-1/2}(\cos\theta_0)\, P_{\mu-1/2}(-\cos\theta)\, Q^{-1-\nu}_{\mu-1/2}(t/r)\,\mu d\mu}{\cos\pi\mu\, P^{1}_{\mu-1/2}(-\cos\theta_0)}$$

$$(t > r) \tag{3.1}$$

Here $Q_{\mu}^{a}(x)$ are the associated Legendre functions of the second kind.

On the basis of the asymptotic properties of these associated Legendre functions as

$|\mu| \to \infty$, it is easy to see that for $2\theta_0 - \theta < \arccos(t/r)$, the integrand in (3.1) for $-r < t < r$ decays exponentially along any ray in the μ-plane outside the real axis; and therefore t the contour L may be deformed into the imaginary axis. Then, since $P^{\alpha}_{\mu-1/2}(x) = P^{\alpha}_{-\mu-1/2}(x)$, the resulting integral vanishes. Thus we shall have everywhere $u = 0$ for $t < 0$, and also for $0 < t < r$, if $2\theta_0 - \theta < \arccos(t/r)$; one obtains the physical flow picture in Fig. 1: outside the bounded region $BCKDN$ there is no disturbance.

If one uses the theorem on residues and reduces the integral in (3.1) to a series corresponding to the roots of the expression $\cos \pi\mu \, P^{1}_{\mu-1/2}(-\cos\theta_0)$, then the series according to roots of the function $\cos \pi\mu$ yields an incident wave with opposite sign (this is easily obtained by applying the inverse Laplace transform to the formula of Sonin (5) on Page 75 of [5] and expressing the Gegenbauer polynomial by means of the associated Legendre functions of the first kind according to Formula (4) on page 177 of [4]). In this way the solution u may be written in the following form:

$$u = \frac{\Gamma(1+\nu)}{2r\cos^2(\theta_0/2)}(r^2 - t^2)^{1/2(1+\nu)} P_0^{-1-\nu}(-t/r) + \qquad (t < r)$$

$$+ \frac{\pi}{2r}\Gamma(1+\nu)(r^2 - t^2)^{1/2(1+\nu)} \sum_{\mu=\mu_n} \frac{(2\mu+1) P_\mu^{1}(\cos\theta_0) P_\mu(-\cos\theta) P_\mu^{-1-\nu}(-t/r)}{\sin \pi\mu \, dP_\mu^{1}(-\cos\theta_0)/d\mu}$$

$$u = \frac{\sin \nu\pi \, \Gamma(1+\nu)}{\pi r \cos^2(\theta_0/2)} (t^2 - r^2)^{1/2(1+\nu)} e^{\pi i\nu} Q_0^{-1-\nu}(t/r) + \frac{\Gamma(1+\nu)}{r} e^{\pi i(1+\nu)} (t^2 - r^2)^{1/2(1+\nu)} \sum_{\mu=\mu_n} \times$$

$$\times \frac{+\sin[(\mu-\nu)\pi] P_\mu^{1}(\cos\theta_0) P_\mu(-\cos\theta) \times (2\mu+1) Q_\mu^{-1-\nu}(t/r)}{\sin \pi\mu \, dP_\mu^{1}(-\cos\theta_0)/d\mu} \qquad (t > r) \qquad (3.2)$$

Here μ_n $(n = 0, 1, 2, \ldots)$ are positive roots of the function $P_\mu^{1}(-\cos\theta)$ (in which $1 < \mu_0 < \mu_1 < \ldots$). It should be noted that the series obtained in (3.2) for $t < r$ and $-1 < \mathrm{Re}\,\nu \leq 0$ is conditionally convergent.

For the surface $\theta = \theta_0$, Formula (3.2) yields

$$u = \frac{\Gamma(1+\nu)}{2r\cos^2(\theta_0/2)}(r^2 - t^2)^{1/2(1+\nu)} P_0^{-1-\nu}(-t/r) - \qquad (t < r)$$

$$- \frac{\Gamma(1+\nu)}{r\sin\theta_0}(r^2 - t^2)^{1/2(1+\nu)} \sum_{\mu=\mu_n} \frac{(2\mu+1) P_\mu^{-1-\nu}(-t/r)}{dP_\mu^{1}(-\cos\theta_0)/d\mu} \qquad (3.3)$$

$$u = \frac{\sin \nu\pi \, \Gamma(1+\nu)}{\pi r \cos^2(\theta_0/2)} e^{\pi i\nu} (t^2 - r^2)^{1/2(1+\nu)} Q_0^{-1-\nu}(t/r) + \qquad (t > r)$$

$$+ \frac{2\Gamma(1+\nu)}{\pi r \sin\theta_0} e^{\pi i\nu} (t^2 - r^2)^{1/2(1+\nu)} \sum_{\mu=\mu_n} \frac{(2\mu+1)\sin[(\mu-\nu)\pi] Q_\mu^{-1-\nu}(t/r)}{dP_\mu^{1}(-\cos\theta_0)/d\mu}$$

As $r \to 0$ we obtain from (3.2)

$$u(0, \theta, t) = \frac{t^\nu}{\cos^2(\theta_0/2)}, \qquad \frac{\partial u}{\partial r} \sim O(r^{\mu_0 - 1}) \qquad (3.4)$$

The value of $u(0, \theta, t)$ thus obtained agrees with the results of (6) (where the value at the vertex of the cone is found without solving the entire problem).

It is easy to verify that the solution (3.2) obtained is such that Laplace and Kontorovich-Lebedev transforms may be applied successively to it, as was assumed initially, and that it is a generalized solution of the wave equation. From estimate (3.4) it follows that the obtained solution (3.2) is unique [2].

Suppose the incident wave has the shape of a step function ($\nu = 0$). The corresponding formulas for this case are obtained from (3.1) to (3.4), when $\nu \to 0$. Studying the solution obtained, it may be shown that the function u undergoes a discontinuity only along the lines BC and DN and that the magnitude of the jump is equal to

$$\left(\frac{\sin(2\theta_0-\theta)}{\sin\theta}\right)^{1/2} \qquad (\theta_0 \leqslant \theta \leqslant 2\theta_0)$$

On the circle $r = t$ the normal derivative $\partial u/\partial r$ is discontinuous. This discontinuity is such that as the arcs AC and DE are approached from either the left or the right, the function $\partial u/\partial r$ becomes logarithmically singular, whereas this function has a finite limit when the arc DKC is approached.

If $\theta_0 \to \pi/2$, then it is easy to show that the solution obtained for $\nu = 0$ tends to the solution for the case of a blunt cone [1]. If the angle θ_0 is small, then a result is obtained which may be found by solving the problem by means of delayed potentials.

Formula (3.3) for the pressure on the cone at $\nu = 0$ has the following form:

$$u(r,\theta_0,t)=\frac{1+x}{2\cos^2(\theta_0/2)}-\frac{\sqrt{1-x^2}}{\sin\theta_0}\sum_{\mu=\mu_n}\frac{(2\mu+1)P_\mu^{-1}(-x)}{dP_\mu^{1}(-\cos\theta_0)/d\mu} \qquad (x<1) \quad (3.5)$$

$$u(r,\theta_0,t)=\frac{1}{\cos^2(\theta_0/2)}+\frac{2\sqrt{x^2-1}}{\pi\sin\theta_0}\sum_{\mu=\mu_n}\frac{(2\mu+1)Q_\mu^{-1}(x)\sin\pi\mu}{dP_\mu^{1}(-\cos\theta_0)/d\mu} \qquad (x>1) \quad (x=t/r)$$

$$u(r,\theta_0,t)\to\frac{1}{\cos^2(\theta_0/2)} \quad \text{as} \quad r\to 0$$

$$u(r,\theta_0,t)\to\frac{1}{\cos^2(\theta_0/2)}-\frac{2}{\pi\sin\theta_0}\sum_{\mu=\mu_n}\frac{(2\mu+1)\sin\pi\mu}{\mu(\mu+1)\,dP_\mu^{1}(-\cos\theta_0)/d\mu} \qquad \text{as } r\to t$$

$$u(r,\theta_0,t)\to 2 \quad \text{as} \quad r\to t/\cos\theta_0$$

Finally, Fig. 3 shows the results of caluclating the function $u(r,\theta_0,t)$ according to Formula (3.5), for the angles θ_0 = 15, 45, and 75°. The roots of the function $P_\mu^{1}(-\cos\theta_0)$ for $\theta_0 = 15°$ were taken from [7] and for $\theta_0 = 45°$ and 75° were calculated by MacDonald's formula [8].

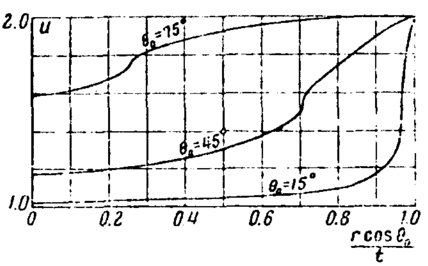

Fig. 3

We note that in the case of incidence of an arbitrary plane wave, the solution is generalized with the help of Duhamel's integral.

4. It is required to find the inverse Laplace transform of the function $I_\mu(pr)/p^\nu$, i.e., to calculate the integral

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{I_\mu(pr)}{p^\nu}e^{pt}\,dp \qquad (4.1)$$

for an arbitrary complex μ and $a > 0$, $r > 0$, $\operatorname{Im} t = 0$, $\operatorname{Re}\nu > -1/2$. The desired integral (4.1) is easily reduced to a sum of two integrals

$$\frac{1}{\pi}\sin\left[\frac{\pi}{2}(\nu-\mu)\right]\int_0^\infty\frac{\sin xt}{x^\nu}J_\mu(xr)\,dx+\frac{1}{\pi}\cos\left[\frac{\pi}{2}(\nu-\mu)\right]\int_0^\infty\frac{\cos xt}{x^\nu}J_\mu(xr)\,dx$$

if one imposes on μ the additional limitation $-1 < \operatorname{Re}(\mu-\nu)$. But each of these integrals, according to Formula 6.699 in [9], may be expressed in terms of a hypergeometric function, and therefore the integral (4.1) is a sum of hypergeometric functions. It turns out that this sum may be expressed by means of associated Legendre functions by Formulas 8.771(2) and 8.775(1) in [9], if one takes into account the error in the index of the hypergeometric function in the second term of the summation in Formula 8.775(1) for $P_\nu^{\mu}(x)$, which should read

$$F\left(\tfrac{1}{2}(\nu+\mu+2),\ \tfrac{1}{2}(\mu-\nu+1);\ \tfrac{3}{2};\ x^2\right)$$

As a result we obtain Formulas

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{I_\mu(pr)}{p^\nu}e^{pt}\,dp=\frac{1}{\sqrt{2\pi r}}(r^2-t^2)^{1/2(\nu-1/2)}P_{\mu-1/2}^{1/2-\nu}(-t/r) \qquad (-r<t<r)$$

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{I_\mu(pr)}{p^\nu}e^{pt}\,dp=\frac{\sqrt{2}}{\pi\sqrt{\pi r}}\sin[(\mu-\nu)\pi]\,e^{\pi i(\nu+1/2)}(t^2-r^2)^{1/2(\nu-1/2)}Q_{\mu-1/2}^{1/2-\nu}(t/r)\quad(t>r) \tag{4.2}$$

(We note that for $t < -r$ the integral (4.1) vanishes.)

By analytic continuation one may show that (4.2) holds in the entire complex μ-plane provided that Re $\nu > -\frac{1}{2}$.

In particular case $\mu - \nu = n$ ($n = 0, 1, 2, \ldots$), Formula (4.2) coincides with known formulas which may be obtained from 29.10 and 29.71 in [10] with the aid of the shift rule (if one considers the lost factor $\frac{1}{2}$ in the right column of Formula 29.71).

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